Investigating functions: Exploring calculus concepts

Stephen Arnold
Australian Catholic University
<s.arnold@signadou.acu.edu.au>

In a previous article in this series, I suggested that it is part of our responsibility as teachers to attempt to induce *perturbations* in our students' mathematical thinking. Especially when teaching seniors and capable students at any level, it is important that we unsettle them, shake their perceptions and attempt, wherever possible, to ensure that they regularly question those things that they take for granted. In this article, we consider some more of my favourite problems: borrowed from a variety of sources, classroom-tested over many years, and always guaranteed to induce discussion, questions and even flashes of inspiration.

The great mathematics educator, Pierre van Hiele, speaks of mathematics teaching and learning in terms of what he calls *structure* and *insight* (1986). Mathematics abounds with structures: patterns and regularities, which form the basis, both for the 'nuts and bolts' work, and also for the wonderful connections which give mathematics its joy and excitement. Van Hiele is a bit coy about defining this term, *structure*, but he is quite clear about *insight*: it is recognition of structure! Consider, for example, teaching students about the difference of two squares. We expect them to acquire a familiarity with this form (this structure), which will enable them to recognise it quickly and intuitively: to know what to do when they encounter it, and to then act appropriately and with purpose.

As a result, few of our senior students have any problem when they encounter questions such as $x^2 - 9$. Their recognition of this structure enables them to respond with a predictable action strategy, resulting in the factorised form $(x - 3)(x + 3)$. The *mathematical objects* that our students encounter tend to elicit such action strategies, to greater and lesser extents. Structures may be *rigid* (producing a strong response, such as the example above); they may also be *feeble*, leaving students unsure of what they should do. Consider the expression $(x - 3)^2 - 9$. In this instance, the recognition of the difference of two squares in this context is likely to be far weaker than the impulse to expand the quadratic; and yet we try to train our students to recognise the subtle as well as the obvious, to demonstrate insight when confronted with mathematical situations.

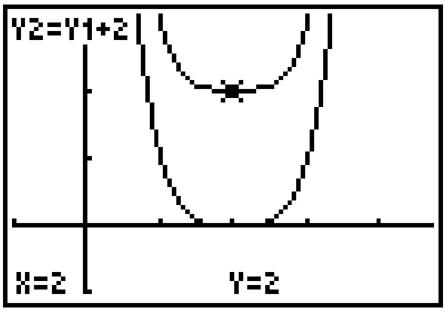
The following problems may help a little in this regard.

1. If you are told that a function $F(x)$ has $F'(2) = 0$ and $F''(2) > 0$, what can you deduce about $F(2)$?
   What does it mean when $f'(x) = f''(x) = 0$?

It is usually helpful to begin with a simpler question, and then lead into a more challenging one. This allows everyone in the class to have the opportunity to at least achieve something, and serves to invite students to 'have a go'. This question, of course, is not difficult at all, but there is a small sting in the tail. There is every likelihood that the response of even our better students to the second question will be of the form: horizontal point of inflexion, since $f'(x) = 0$ defines a horizontal point (gradient zero) while $f''(x) = 0$ indicates a point of inflexion, provided the sign of $f''(x)$ changes (change of concavity).

Ask them then to consider functions such as x^4, x^6 and so on. A better response would be that, while $f'(x) = 0$ does define a horizontal point, $f''(x) = 0$ *suggests* a point of inflexion, it does not *define* one. Other than in simple cases, such as quadratics and cubics, further work should always be done to ascertain the behaviour of the function at such a point.

A suitable further investigation would involve students attempting to find other examples where the double negative does *not* lead to the expected horizontal point of inflexion: this should lead them to conclude that the problem certainly occurs wherever multiple roots of an even power occur. Ensure that they appreciate, however, that not only would $(x - 2)^4$ behave in such a way, but so would $(x - 2)^4 + 2$!



2. Suppose transportation specialists have determined that $G(v)$, the number of litres of fuel per hundred kilometres that a vehicle consumes, is a function of the speed of the vehicle, in kilometres per hour.
   Interpret, in terms of fuel consumption, the finding that $G'(110) = 0.4$.
   How might this finding be used in a debate about setting an appropriate national speed limit? (Heid, 1988)

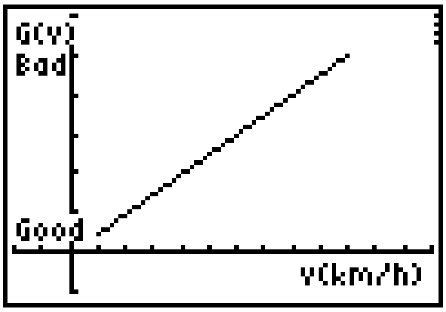
This classic problem (and the one which follows) were developed for one of the very early ground-breaking research studies on the effects of the use of computer algebra systems (CAS) upon student understanding and acquisition of skills. The problems generated for the project remain, I believe, among the best of their kind in assessing student's deeper appreciation of the tools of calculus.

The original question, originating in the United States, used miles per hour for vehicle speed, and miles per gallon for the measure of fuel consumption (with the finding, $G'(55) = 0.4$). In attempting to make the problem appropriate for our metric-minded students, the units of speed became kilometres per hour, and those for fuel consumption became litres per hundred kilometres. This conversion causes something of a problem, since it actually *inverts* the function. While good fuel consumption in miles per gallon involves big numbers (the higher the mpg, the better the fuel economy), when using the metric units the opposite is true: the less litres one uses to travel 100 kilometres, the better!

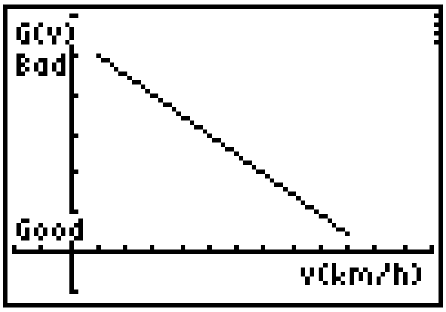
One way to begin this problem could be to give students both forms (metric and imperial), explain that we are unsure of whether our conversion of the problem is correct, and ask them to verify that the finding should convert from $G'(55) = 0.4$ into $G'(110) = 0.4$ (and not $G'(110) = -0.4$!) It is easier, in fact, to produce a model using the imperial units, while the verification of the result involves students in a valid and realistic mathematical enquiry.

Traditionally, however, I have not shared my uncertainties regarding my conversion of the problem: I have given it to them as stated, and suggested that their primary task is to attempt to produce a graph which realistically models fuel consumption. This is most appropriate for students in the senior years, since they are generally beginning their driving experience.

A class, which may not have done much algebraic and graphical modelling, will need a little help getting started. First, they need to understand that this question is not about finding algebraic formulas or numerical values: it is far more intuitive and visual, We are looking for an approximate curve shape, which offers a reasonable model. My advice: begin with $y = x$ and see in what ways this matches or fails to match the situation.

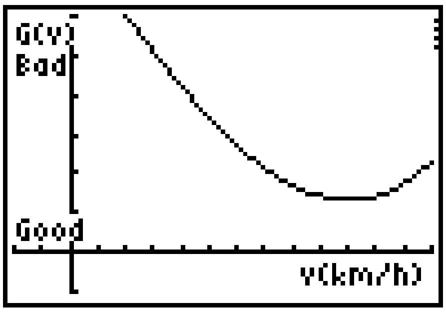


In this scenario, it is helpful, once the axes are drawn as shown, to add the words ‘good’ and ‘bad’ along the vertical (fuel consumption) axis. In litres per 100 kilometres, high values are ‘bad’, low values, ‘good’. Students should quickly realise that for normal driving conditions, driving at low speed is ‘bad’ for fuel consumption; driving at moderate speeds, ‘good’. This suggests a negative gradient, rather than a positive one, and a starting model like that shown.



The next step would be to examine conditions at the extremes: a straight-line model fails, since it suggests a certain value at zero velocity [Why is this impossible?], and a certain velocity when fuel consumption would be zero! We need to curve our model.

This should lead (eventually, hopefully after a few arguments!) to a far more realistic model, as shown. Likely to be asymptotic at $v = 0$ (why?) and suggesting an optimum velocity, after which fuel economy again degrades. If this is taken to be 100 kilometres per hour [Why? Discuss this carefully, using students’ own experiences and observations] then the finding concerning the first derivative is, in fact, supported: just after this optimum value, the gradient is small but positive, suggesting a turning point just passed!



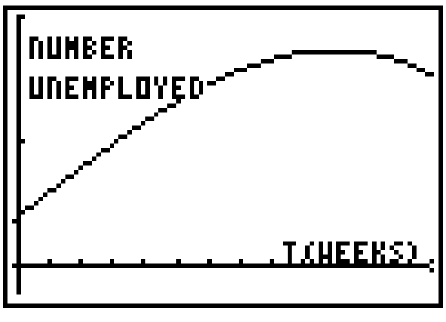
3. $E(t)$ represents the number of people unemployed in a country, t weeks after the election of a new government. Translate each of the following facts about the graph, $y = E(t)$, into statements about the unemployment situation:
   a. The y-intercept of $y = E(t)$ is 2 000 000.
   b. $E(20) = 3\ 000\ 000$
   c. The slope of $y = E(t)$ at $t = 20$ is 10 000.
   d. $E''(36) = -800$ and $E'(36) = 0$. (Adapted from Heid, 1988)

Once again, students are challenged to reassess the 'obvious'. Such concepts as y-intercept and slope, and even first and second derivatives may be no problem in the usual graphical context, but when applied to the real world, they take on much greater meaning and significance for our students.

I usually frame this problem within a scenario within which students are reporters for a newspaper or television news service, and they are required to write a short feature article evaluating the government's economic strategies since their election, using the information given. This implies, of course, that they must translate the mathematics into words that they and other less mathematical readers will understand.

While they will have few problems with the first two statements, teachers should ensure that they come to grips with the third statement in terms of what the gradient actually represents here (number of people becoming unemployed every week).

Finally, of course, their interpretation of the first and second derivative information is critical in terms of their understanding of the scenario: if they are not careful, they will write a very critical article of the government's policies when, in reality, the data clearly indicates that the unemployment rate was slowing significantly at 20 weeks (compare the average rate of change over the first 20 weeks — an additional one million people unemployed over 20 weeks is a rate of 50 000 people per week — with the instantaneous rate of change given by the slope of 10 000 at $t = 20$).



Beginning once again with $y = x$, students should arrive at a graphical model similar to that shown, and a greater appreciation of the concepts involved.

Our final problem set was sent some years ago to the AAMT email list (aamt-l@edna.edu.au) by Evan Romer, a teacher at Susquehanna Valley High School, New York state. This email offered a delightful collection of eight problems related to understanding of the second derivative. All were sourced from newspaper articles and each illustrates both understandings and misunderstandings which surround rates of change. I have included two here.

4(i) **Decline in CFCs Signals Changes in Ozone Depletion**

The Washington Post.

The amount of ozone-destroying CFCs in the atmosphere, which had been rising rapidly for decades, suddenly slowed its rate of increase in 1989 and has nearly levelled off since then, scientists at the National Oceanic and Atmospheric Administration has found.

Chlorofluorocarbon concentrations (parts per trillion).

1977–1994: Observed concentrations (NOAA)

1995–2000: Calculated concentrations (Dupont)

If C represents the concentration of CFCs in the atmosphere as a function of time, then in 1994:

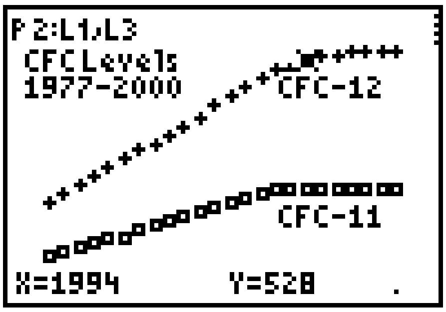
(a) C was positive/negative/zero/cannot tell?

(b) C' was positive/negative/zero/cannot tell?

(c) C'' was positive/negative/zero/cannot tell?

(d) Is the headline accurate?

[The article was accompanied by a graph: I have recreated it here from a table provided]



4(ii) [From a 1993 newspaper article]

IBM Corp. is starting to pull out of the red, company officials said Tuesday. The computer maker took a loss of 12 cents a share in the third quarter. That compares with a per-share loss of $4.87 a year ago…

If P represents IBM's net profit per share, then

(a) C was positive/negative/zero/cannot tell?

(b) C' was positive/negative/zero/cannot tell?

(c) C'' was positive/negative/zero/cannot tell?

There is much food for thought in such questions, and much value in asking our students to think critically about what they read in the news. Sadly, not all who attempt to interpret mathematics do so correctly: but who is usually going to notice?

[I have created small programs as a StudyCards stack of problems for use with TI-83Plus graphic calculators in support of some of these and other problems. Teachers who would like to use these are invited to email me at s.arnold@signadou.acu.edu.au and I will send them on.]

References

Heid, M. K. (1988). Resequencing skills and concepts in applied calculus using the computer as a tool. *Journal for Research in Mathematics Education, 19*(1), 3–25.

van Hiele, P. (1986). *Structure and Insight: A Theory of Mathematics Education.* Orlando, FA: Academic Press.